

A theoretical study of the vibration and acoustics of ancient Chinese bells

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In this paper, the acoustics of an ancient Chinese bell, which was made some 3000 years B. C., is studied theoretically. In ancient times, a set of the bells was used as a musical instrument. Unlike a western church bell and an ancient Indian bell, an ancient Chinese bell has two interesting acoustics. First, two tones can be heard separately as the bell is struck at two special points. The interval between the two pitches is always a minor or major third. Second, tones of the bell attenuate quickly, which is necessary for a musical instrument. So, an ancient Chinese bell is sometimes called a two-tone bell or a music bell. Although a three-dimensional model should be used to simulate the acoustics of the bell, a simplified model proposed in this paper does give some insight. Based on the lens-shaped cross section of an ancient Chinese bell, two tones of an ancient Chinese bell can be simulated by the vibration of a double-circular arch and the quick attenuation of tones can be simulated by acoustics of a cylinder with the lens-shaped cross section like a double-circular arch. Numerical results on the vibration and acoustics of the models are presented. © 2003 Acoustical Society of America. [DOI: 10.1121/1.1600727]

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I. INTRODUCTION

In the 1950s and 1970s, two complete sets of ancient Chinese bells were discovered in *Henan* province and *Hubei* province in China, respectively. A sketch of the bell is shown in Fig. 1.

These ancient bells have two interesting acoustics. First, two tones can be heard separately as two special points on the bell are struck. The two special points, inscribed on the bell as shown in Fig. 1, are called *SUI* and *GU* (*SUI* and *GU* are pronunciation of two words in ancient Chinese language), respectively. The *SUI* pitch is a minor or major third lower than *GU* pitch. Second, the tones of the bell attenuate quickly, without prolonged echo. This is called short duration of sound. Sinyang¹ described in detail the acoustics of the bell.

Unlike a western church bell or an ancient Indian bell, the cross section *a-a* of an ancient Chinese bell is lens-shaped or eye-shaped as shown in Fig. 1. Why is the cross section not round? The earliest record on short duration of sound of a Chinese bell can be found in an ancient Chinese book *Mengxi Bitan* (Dream Stream Essays) written by *Shen Kuo* (an ancient Chinese statesman and scientist, 1030–1095, Northern Song Dynasty). Through observation, *Shen Kuo* noted that the tone of a round bell has more prolonged echo than that of an oblate bell, and that an oblate bell is suitable for a musical instrument. In the view of modern science, the acoustics of the ancient Chinese bell is determined by its special structure.

Sinyan¹ studied an ancient Chinese bell experimentally. Rossing *et al.*² investigated vibration modes of a Chou Dy-

nasty bell by means of holographic interferometry and by scanning the sound field near the bell.

Theoretically, a three-dimensional model should be used to simulate acoustics of the bell. To the best of our knowledge, there is no such report on a theoretical study of the bell. Because a three-dimensional model is very complex, a simplified model is proposed in this paper to investigate the acoustics of the bell. Based on the lens-shaped section of the bell, it is supposed that free vibration of cross section of an ancient Chinese bell can be simulated by the free vibration of a double-circular arch as shown in Fig. 2. Interests in the natural vibration of arch can be traced back to the 19th century. Chidamparam³ gave a survey on vibration of an arch. In this paper, the effects of shear deformation and rotary inertia in vibration are neglected. By extensional theory of planar

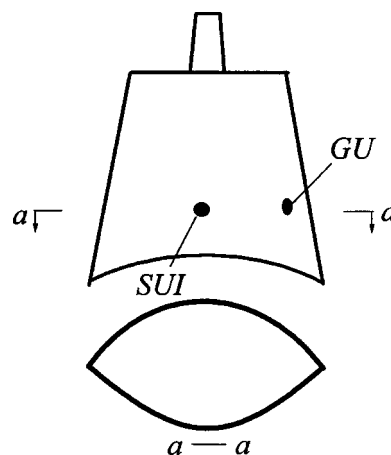


FIG. 1. Sketch of an ancient Chinese bell. There are two striking points *SUI* and *GU* corresponding to two tones. Tones of the bell attenuate quickly, without prolonged echo. The cross section *a-a* of the bell is lens-shaped, which is similar to a double-circular arch.

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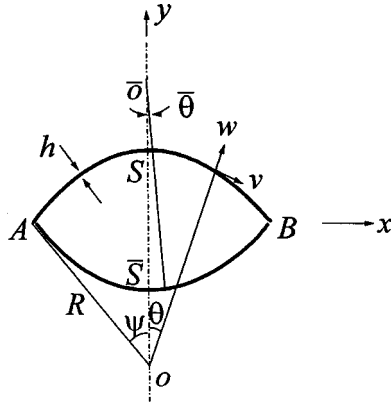


FIG. 2. A double-circular arch is made of two circular arches ASB and ASB $\bar{}$ with radius R . h denotes thickness of each arch, ψ denotes a half of central angle of each arch, and θ denotes coordinate along the arch ASB. w denotes the normal displacement of centroidal axis, while v denotes the tangential component. Symbols with an upper bar “ $\bar{}$ ” denote parameters on the arch ASB $\bar{}$.

curved beam, the basic vibration equations for a circular arch have been obtained by Chidamparam³ and Soedel.⁴ Exact solution of the basic vibration equations, which Chidamparam⁵ studied, is used to analyze the vibration of the double-circular arch. Furthermore, it is supposed that short duration of sound can be simulated by acoustics of a vibrating cylinder with the lens-shaped cross section like a double-circular arch. According to Morse,⁶ we can get the total power radiated in an unit time from the vibrating cylinder, which is the source of sound field. Then, we get attenuation exponential of sound to explain the short duration of sound.

II. VIBRATION OF A DOUBLE-CIRCULAR ARCH

A. Exact solution of natural mode on planar circular arch

Because the cross section of an ancient Chinese bell is lens-shaped or eye-shaped, we suppose that the shape of the cross section is a double-circular arch which is made of a circular arch ASB and a circular ASB $\bar{}$, each with radius R as shown in Fig. 2. Here, h denotes thickness of each arch, ψ denotes a half of the central angle of each arch, ρ denotes mass density, E denotes Young's modulus, and θ denotes angular coordinate along arch ASB, w denotes the normal displacement along centroidal axis, while v denotes tangential displacement. In Fig. 2, symbols with an upper bar “ $\bar{}$ ” denote parameters on arch ASB $\bar{}$.

In extensional theory, the dynamic equations for a circular arch undergoing planar vibration are Eqs. (12a), (12b) in Chidamparam³ as

$$\begin{aligned} (1+k^2) \frac{\partial^2 v}{\partial \theta^2} + \left(-k^2 \frac{\partial^3}{\partial \theta^3} + \frac{\partial}{\partial \theta} \right) w &= \frac{\rho R^2}{E} \frac{\partial^2 v}{\partial t^2} \\ \left(-k^2 \frac{\partial^3}{\partial \theta^3} + \frac{\partial}{\partial \theta} \right) v + \left(k^2 \frac{\partial^4}{\partial \theta^4} + 1 \right) w &= -\frac{\rho R^2}{E} \frac{\partial^2 w}{\partial t^2}, \end{aligned} \quad (1)$$

where $k^2 = h^2/12R^2$.

For the circular arch ASB, let

$$v = V(\theta)e^{j\omega t}, \quad w = W(\theta)e^{j\omega t}, \quad (2)$$

where V and W are the amplitudes of tangential and normal vibration modes, respectively, and ω is the natural frequency.

Substituting (2) into (1), we have

$$\begin{aligned} \left[(1+k^2) \frac{d^2}{d\theta^2} + \frac{\rho R^2}{E} \omega^2 \right] V + \left(-k^2 \frac{d^3}{d\theta^3} + \frac{d}{d\theta} \right) W &= 0 \\ \left(-k^2 \frac{d^3}{d\theta^3} + \frac{d}{d\theta} \right) V + \left(k^2 \frac{d^4}{d\theta^4} + 1 - \frac{\rho R^2}{E} \omega^2 \right) W &= 0. \end{aligned} \quad (3)$$

From (3)

$$\Delta V = 0, \quad (4)$$

$$\Delta W = 0, \quad (5)$$

where Δ is an operator

$$\begin{aligned} \Delta = \left[(1+k^2) \frac{d^2}{d\theta^2} + \frac{\rho R^2}{E} \omega^2 \right] \left(k^2 \frac{d^4}{d\theta^4} + 1 - \frac{\rho R^2}{E} \omega^2 \right) \\ - \left(-k^2 \frac{d^3}{d\theta^3} + \frac{d}{d\theta} \right)^2. \end{aligned} \quad (6)$$

Letting $W(\theta) = e^{\lambda\theta}$, we have the character equation of Eqs. (4) and (5)

$$\lambda^6 + a_2 \lambda^4 + a_1 \lambda^2 + a_0 = 0, \quad (7)$$

where

$$\begin{aligned} a_0 &= \frac{1}{k^2} \frac{\rho R^2}{E} \omega^2 \left(1 - \frac{\rho R^2}{E} \omega^2 \right), \\ a_1 &= 1 - \left(1 + \frac{1}{k^2} \right) \frac{\rho R^2}{E} \omega^2, \\ a_2 &= 2 + \frac{\rho R^2}{E} \omega^2. \end{aligned} \quad (8)$$

Equation (7) is a cubic equation in λ^2 . So, we have a set of roots in (7) for different ω . For a different set of roots in (7), we have the exact solution of (5) as

$$W(\theta) = \mathbf{c}^T \mathbf{f}, \quad (9)$$

where $\mathbf{c}_{6 \times 1}$ is a vector with six unknown constants for arch ASB and $\mathbf{f}_{6 \times 1}$ is solution vector in (5) for a set of roots of (7). Similarly, we have the exact solution V in (4)

$$V(\theta) = \tilde{\mathbf{c}}^T \mathbf{f}, \quad (10)$$

where $\tilde{\mathbf{c}}_{6 \times 1}$ is a vector with six other unknown constants for arch ASB and $\mathbf{f}_{6 \times 1}$ is the same solution vector as that in Eq. (9).

Let

$$\frac{d\mathbf{f}}{d\theta} = \mathbf{L}\mathbf{f}, \quad (11)$$

where $\mathbf{L}_{6 \times 6}$ is a matrix. Substituting (9), (10), (11) into the first equation in (3), we have the relation between $\mathbf{c}$ and $\tilde{\mathbf{c}}$

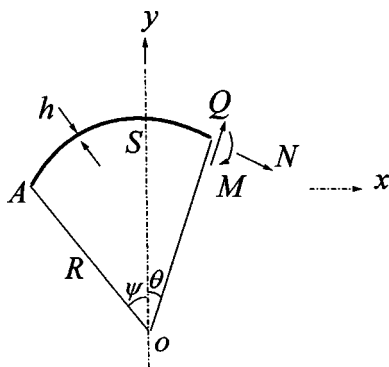


FIG. 3. N , M , and Q denote mode of internal forces: axial tension, moment, and shear force, respectively, of the arch ASB.

$$\tilde{\mathbf{c}}^T = \mathbf{c}^T \mathbf{G}, \quad (12)$$

where $\mathbf{G}_{6 \times 6}$ is a matrix

$$\mathbf{G} = (k^2 \mathbf{L}^3 - \mathbf{L}) \left((k^2 + 1) \mathbf{L}^2 + \frac{\rho R^2}{E} \omega^2 \mathbf{I} \right)^{-1}, \quad (13)$$

where $\mathbf{I}_{6 \times 6}$ is unit matrix. Substituting (12) into (10), we have

$$V(\theta) = \mathbf{c}^T \mathbf{G} \mathbf{f}. \quad (14)$$

Equations (9) and (14) are the exact solution of the normal and tangential component of the natural mode, respectively, for the circular arch ASB.

The internal forces of the circular arch ASB are shown in Fig. 3, where N , M , and Q denote mode of axial tension, moment, and shear force, respectively. The relations between these internal forces and displacements are

$$\begin{aligned} N &= \frac{Eh}{R} \left(\frac{dV}{d\theta} + W \right), \\ M &= \frac{Eh^3}{12R^2} \left(\frac{dV}{d\theta} - \frac{d^2W}{d\theta^2} \right), \\ Q &= \frac{Eh^3}{12R^3} \left(\frac{d^2V}{d\theta^2} - \frac{d^3W}{d\theta^3} \right). \end{aligned} \quad (15)$$

Similarly, we have the tangential and normal component of natural mode for the circular arch ASB

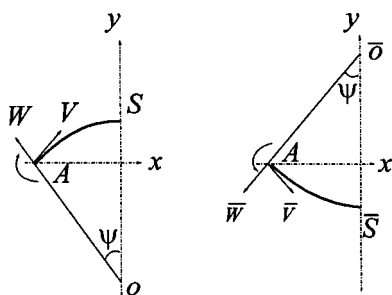


FIG. 4. Displacements and rotation at point A of the arch ASB and the arch ASB. V and W denote, respectively, tangential and normal vibration mode of the arch ASB. $\bar{V}$ and $\bar{W}$ denote, respectively, tangential and normal vibration mode of the arch ASB. Displacements and rotation at point A are continuous as the double-circular arch vibrates.

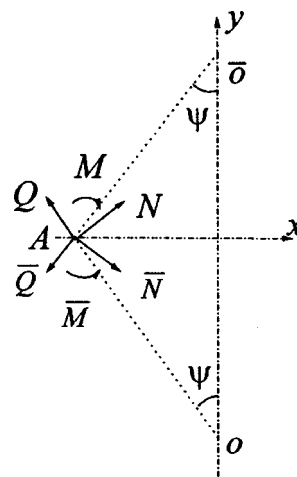


FIG. 5. Internal forces at point A of the arch ASB and the arch ASB are equilibrium in x , y , and rotation in direction perpendicular to x and y direction as the double-circular arch vibrates.

$$\bar{\mathbf{V}} = \bar{\mathbf{c}}^T \mathbf{G} \mathbf{f}, \quad (16)$$

$$\bar{\mathbf{W}} = \bar{\mathbf{c}}^T \mathbf{f}, \quad (17)$$

where $\bar{\mathbf{c}}_{6 \times 1}$ is a vector with six unknown constants for arch ASB, $\mathbf{f}_{6 \times 1}$ is the same solution vector as that in Eq. (9), and $\mathbf{G}_{6 \times 6}$ is given in Eq. (13).

Expressions for N , M , and Q , which denote the axial tension, moment, and shear force, respectively, of the mode in the circular arch ASB, are the same as (15).

B. Continuity conditions between the circular arch ASB and circular arch ASB

Referring to Fig. 4, displacements and rotation angle at point A of arch ASB and arch ASB are continuous. Continuity conditions of displacement are

$$-W \sin \psi + V \cos \psi = -\bar{W} \sin \psi + \bar{V} \cos \psi \quad \text{in } x \text{ direction}, \quad (18)$$

$$W \cos \psi + V \sin \psi = -\bar{W} \cos \psi - \bar{V} \sin \psi \quad \text{in } y \text{ direction}, \quad (19)$$

$$V - \frac{dW}{d\theta} = - \left(\bar{V} - \frac{d\bar{W}}{d\theta} \right) \quad \text{in rotation}, \quad (20)$$

where $\theta = \bar{\theta} = -\psi$. Similarly, continuity conditions of displacements at point B are

$$W \sin \psi + V \cos \psi = \bar{W} \sin \psi + \bar{V} \cos \psi, \quad (21)$$

$$W \cos \psi - V \sin \psi = -\bar{W} \cos \psi + \bar{V} \sin \psi, \quad (22)$$

$$V - \frac{dW}{d\theta} = - \left(\bar{V} - \frac{d\bar{W}}{d\theta} \right), \quad (23)$$

where $\theta = \bar{\theta} = \psi$.

Referring to Fig. 5, internal forces at point A of arch ASB and arch ASB are equilibrium. Continuity conditions of force are

$$N \cos \psi - Q \sin \psi + \bar{N} \cos \psi - \bar{Q} \sin \psi = 0$$

in x direction, (24)

$$N \sin \psi + Q \cos \psi - \bar{N} \sin \psi - \bar{Q} \cos \psi = 0$$

in y direction, (25)

$$M - \bar{M} = 0 \quad \text{in moment,} \quad (26)$$

where $\theta = \bar{\theta} = -\psi$. Similarly, continuity conditions of force at point B are

$$-N \cos \psi - Q \sin \psi - \bar{N} \cos \psi - \bar{Q} \sin \psi = 0, \quad (27)$$

$$N \sin \psi - Q \cos \psi - \bar{N} \sin \psi + \bar{Q} \cos \psi = 0, \quad (28)$$

$$M - \bar{M} = 0, \quad (29)$$

where $\theta = \bar{\theta} = \psi$.

Equations (18)–(29), in total 12 equations, are the continuity conditions at points A and B between arch ASB and arch ASB. Substituting Eqs. (9), (14), (16), and (17) into (18)–(29), and taking (15) into account, we have

$$\mathbf{H}(\omega) \begin{pmatrix} \mathbf{c} \\ \mathbf{c} \end{pmatrix} = \mathbf{0}, \quad (30)$$

where $\mathbf{H}(\omega)_{12 \times 12}$ is a matrix.

C. Natural frequencies and natural modes of a double-circular arch

As the double-circular arch vibrates, $(v, w) \neq \mathbf{0}$ in (1) for arch ASB and arch ASB. So, from (2), we have

$$(W, V, \bar{W}, \bar{V}) \neq \mathbf{0}, \quad (31)$$

and considering the expression for $W, V, \bar{W}, \bar{V}$ in Eqs. (9), (14), (16), and (17), we have

$$\begin{pmatrix} \mathbf{c} \\ \mathbf{c} \end{pmatrix} \neq \mathbf{0}. \quad (32)$$

Considering (32) and (30), we have

$$\det \mathbf{H}(\omega) = 0. \quad (33)$$

The roots $\omega \in (0, +\infty)$ of (33) are the natural frequencies of the double-circular arch in Fig. 2.

Equation (33) is a transcendental equation in which ω is an unknown quantity. In this paper, these roots in $(0, +\infty)$ are obtained numerically by the bisection method (see Sec. II D). Substituting a root of (33) into (30), (9), (14), (16), and (17), the natural mode corresponding to this root can be found.

D. Calculation of $\det \mathbf{H}(\omega)$

In the bisection method for searching the roots of (33), it is necessary to calculate value of $\det \mathbf{H}(\omega)$ for a given ω . The process for calculating $\det \mathbf{H}(\omega)$ is as follows: First, for a given ω , we can get the analytical solution of λ^2 in (7). Second, we can get the six roots λ of (7). Third, according to the six roots, we can get analytical expression of solution vector $\mathbf{f}$ in (9) for differential equation (4) or (5). Fourth, we have an expression of general solution (9), (14), for (4), (5), respectively. Fifth, substituting expressions for V and W into

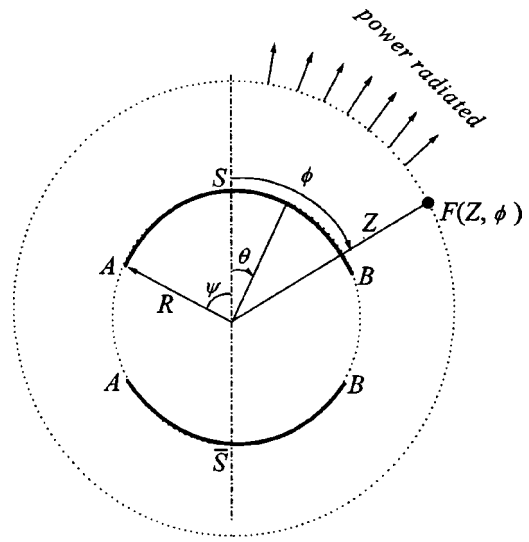


FIG. 6. Model of sound attenuation. As the cylinder with the lens-shaped cross section like a double-circular arch vibrates, its moving surface compresses air periodically to generate a sound wave and radiates acoustic power outward.

(15), we have an expression for the internal forces. For arch ASB, we repeat the process above. Sixth, substituting the analytical expressions for displacements and internal forces into (18)–(29), we have an analytical expression of matrix $\mathbf{H}(\omega)_{12 \times 12}$ in (30) for a given ω . Finally, we have value of determinant $\det \mathbf{H}(\omega)$ numerically.

III. SOUND ATTENUATION OF A CYLINDER WITH THE LENS-SHAPED CROSS SECTION LIKE A DOUBLE-CIRCULAR ARCH

A. The total power radiated in an unit time

In fact, structure of an ancient Chinese bell is relatively complicated.¹ To analyze its acoustics, we have to do some simplifications. First, since the structure of an ancient Chinese bell in Fig. 1 is approximately a cylinder with a lens-shaped cross section, we consider an infinitely long cylinder with the lens-shaped cross section like a double-circular arch to model the bell's structure. Second, surface vibration of the cylinder has the same natural frequencies and modes as the double-circular arch in Fig. 2. Third, in this treatment, the viscosity of air is neglected. Fourth, we suppose that the sound wave is a cylindrical wave so that the problem is two-dimensional.

As the cylinder vibrates, its moving surface compresses air periodically to generate a sound wave, and it radiates acoustic power outward as shown in Fig. 6. Using polar coordinates (Z, ϕ) , the sound field far away from the source is considered where $Z \gg R$.

In Fig. 6, θ denotes coordinate of ASB. According to (2), the radial velocity at point θ of ASB is $LW(\theta)j\omega e^{j\omega t}$, in which L is the displacement amplitude and $W(\theta)$ is the normal component of the natural mode of ASB. The radial velocity at point θ of ASB is $LW(\theta)j\omega e^{j\omega t}$, where L is the displacement amplitude and $W(\theta)$ is the normal component of the natural mode of ASB. ρ_0 and c denotes the density of air and the velocity of sound in air, respectively, according to

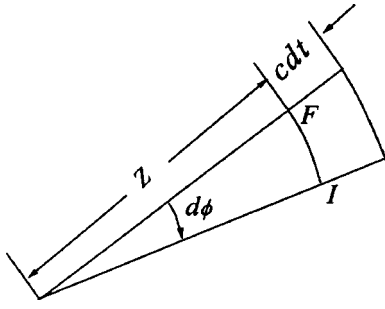


FIG. 7. In an infinitesimal time interval dt , power radiated through surface $F-I$ equals kinetic energy of air particles in square cdt by $Zd\phi$, where c is sound velocity.

Eq. (7.3.10) on page 363 in Morse.⁶ In Fig. 6, the acoustic pressure p and the velocity u of an air particle at $F(Z, \phi)$ far away from the sound source ASB and ASB are

$$p(Z, \phi, t) = \rho_0 c \sqrt{\frac{R}{Z}} e^{j\delta(Z-ct)} \left[\int_{-\psi}^{\psi} f(\phi - \theta) L W(\theta) j \omega d\theta + \int_{\pi-\psi}^{\pi+\psi} f(\phi - \theta) L \bar{W}(\pi - \theta) j \omega d\theta \right], \quad (34)$$

$$u(Z, \phi, t) = \frac{p}{\rho_0 c}, \quad (35)$$

where $\delta = \omega/c$ and f is shown in Eq. (7.3.8) on page 361 in Morse⁶

$$f(\lambda) = \sqrt{\frac{2}{\pi^3 \delta R}} \sum_{n=0}^{\infty} \frac{\cos(n\lambda)}{E_n} \exp \left\{ -j \left[\gamma_n + \frac{\pi}{4} (2n+1) \right] \right\}, \quad (36)$$

where E_n and γ_n are determined by

$$J_1(\delta R) = \frac{1}{2} E_0 \sin \gamma_0, \quad N_1(\delta R) = -\frac{1}{2} E_0 \cos \gamma_0, \quad (37)$$

$$J_{n+1}(\delta R) - J_{n-1}(\delta R) = 2 E_n \sin \gamma_n, \quad (38)$$

$$N_{n-1}(\delta R) - N_{n+1}(\delta R) = 2 E_n \cos \gamma_n \quad (n = 1, 2, 3, \dots),$$

where $J_n()$ and $N_n()$ are the Bessel and Neumann function of the n th order, respectively.

Substituting (34) into (35), the velocity of an air particle at $F(Z, \phi)$ is

$$u(Z, \phi, t) = j \omega L \sqrt{\frac{R}{Z}} e^{j\delta(Z-ct)} \left[\int_{-\psi}^{\psi} f(\phi - \theta) W(\theta) d\theta + \int_{\pi-\psi}^{\pi+\psi} f(\phi - \theta) \bar{W}(\pi - \theta) d\theta \right]. \quad (39)$$

From Eq. (39), we have the maximum velocity of an air particle at $F(Z, \phi)$

$$u_{\max}(Z, \phi) = \omega L \sqrt{\frac{R}{Z}} \left[\int_{-\psi}^{\psi} f(\phi - \theta) W(\theta) d\theta + \int_{\pi-\psi}^{\pi+\psi} f(\phi - \theta) \bar{W}(\pi - \theta) d\theta \right]. \quad (40)$$

Let's consider the movement of air particles near surface $F-I$ as the sound wave is traveling through $F-I$ in Fig. 7.

Before the sound wave reaches $F-I$, air particles near $F-I$ are still. At time t , sound wave reaches $F-I$. After an infinitesimal time interval dt , i.e., at time $t+dt$, sound wave travels through $F-I$ and air particles in volume $Zc d\phi dt$ get vibrating with maximum velocity $u_{\max}$. The kinetic energy of these air particles equals the power radiated through surface $F-I$ in time interval dt . So, power radiated through surface $F-I$ in an infinitesimal time interval dt is

$$\frac{1}{2} \rho_0 Z c d\phi dt u_{\max}^2. \quad (41)$$

Substituting (40) into (41), and integrating (41) with respect to ϕ from 0 to 2π , we have the total power radiated through the large circle with radius Z in Fig. 6 in unit time

$$\frac{d\Pi}{dt} = R \omega^2 L^2 \Sigma, \quad (42)$$

where

$$\Sigma = \frac{1}{2} \rho_0 c \int_0^{2\pi} \left[\int_{-\psi}^{\psi} f(\phi - \theta) W(\theta) d\theta + \int_{\pi-\psi}^{\pi+\psi} f(\phi - \theta) \bar{W}(\pi - \theta) d\theta \right]^2 d\phi, \quad (43)$$

and $d\Pi$ is the acoustic power radiated through the circle with radius Z in time interval dt . Expression (42) is the total power radiated in a unit time from the cylinder with the lens-shaped cross section like a double-circular arch.

B. Attenuation exponential of the vibrating cylinder with the lens-shaped cross section

The kinetic energy of the vibrating cylinder with the lens-shaped cross section like a double-circular arch is

$$T = \frac{1}{2} R \omega^2 L^2 \Omega, \quad (44)$$

where

$$\Omega = \rho h \left[\int_{-\psi}^{\psi} (W^2 + V^2) d\theta + \int_{\pi-\psi}^{\pi+\psi} (\bar{W}^2 + \bar{V}^2) d\theta \right], \quad (45)$$

where ρ is density of the cylinder. By neglecting losses of energy by all means except acoustic radiation, we have

$$dT + d\Pi = 0, \quad (46)$$

where dT is the decrease of vibration energy of the cylinder in dt . Substituting (42) and (44) into (46), we have

$$\frac{L}{L_0} = e^{-\beta t}, \quad (47)$$

TABLE I. Natural frequency (Hz) of a circular ring. As $\psi = \pi/2$, the double-circular arch becomes a circular ring. Numerical results of frequencies in this paper are compared with analytical results shown in Soedel (Ref. 4) and Timoshenko (Ref. 7).

(Hz)	Analytical solution		Numerical solution This paper
	Soedel (Ref. 4)	Timoshenko (Ref. 7)	
f_1	126.306	126.301	126.27
f_2	357.248	357.229	357.24

TABLE II. Parameters of two kinds of double-circular arches with different total arc length. The total length is sum of arc length ASB and ASB in Fig. 2.

Double-circular arch	Density (kg/m ³) ρ	Young's modulus (GPa) E	Thickness (mm) h	Total length (mm)
No. 1	7.85×10^3	200.6	2.0	$2 \times 3.14 \times 100.0$
No. 2			4.0	$2 \times 3.14 \times 50.5$

where L and L_0 are the displacement amplitude of the cylinder at present time t and initial time $t=0$, respectively, and where

$$\beta = \frac{\Sigma}{\Omega} \quad (48)$$

is the attenuation exponential of sound.

IV. NUMERICAL RESULTS

A. Natural frequencies for a circular ring

As $\psi = \pi/2$, the double-circular arch in Fig. 2 becomes a circular ring. As Young's modulus $E = 200.6$ GPa, density $\rho = 7.85 \times 10^3$ kg/m³, thickness $h = 2 \times 10^{-3}$ m, and radius $R = 1 \times 10^{-1}$ m, the numerical results in this paper on the first and the second natural frequencies are compared with the results in Soedel⁴ and Timoshenko⁷ in Table I.

B. Two tones of an ancient Chinese bell

Parameters of two kinds of double-circular arches No. 1 and No. 2 in different total arc length are listed in Table II. The first and the second natural modes of a double-circular arch are shown in Fig. 8. Ratios of the second frequency f_2 to the first frequency f_1 of the two kinds of double-circular arches are listed in Table III for different ψ . The numerical results indicate that values of f_2/f_1 for the two kinds of double-circular arches No. 1 and No. 2 remain the same for a given ψ as shown in Table III.

In Fig. 8 and Table III, we can see that

- (i) As the central point (*SUI* position on the bell) is struck, the double-circular arch vibrates at the first frequency f_1 . As the side point (*GU* position on the bell) is struck, the double-circular arch vibrates at the second frequency f_2 .

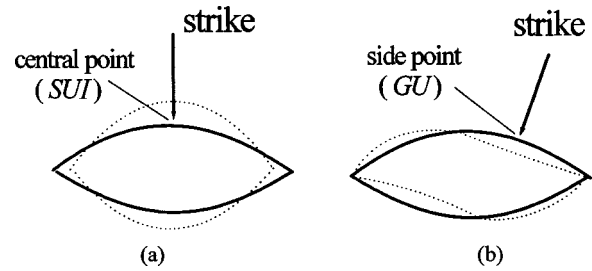


FIG. 8. Two tones of an ancient Chinese bell. As the central point (*SUI* position on bell) is struck, the double-circular arch vibrates in the first frequency with the first natural mode in (a). As the side point (*GU* position on bell) is struck, it vibrates in the second frequency with the second natural mode in (b).

- (ii) If the cross section is circular, the bell emits simply one pitch as an arbitrary point is struck. If the cross section is lens-shaped, the bell emits two pitches as *SUI* position and *GU* position are struck, respectively.

C. Short duration of sound in Chinese bell

For example, we take the cylinder with the lens-shaped cross section like a double-circular arch No. 2 in Table II to simulate the short duration of sound in Chinese bell. Under 20 °C, density of air is $\rho_0 = 1.293$ kg/m³ and sound velocity in air is $c = 340$ m/s. Attenuation exponentials β in (47) for the first tone and the second tone are shown in Table IV for different ψ .

In Table IV, we can see that: The smaller the angle ψ is, the bigger the attenuation exponential β is. In other words, the more oblate the bell is, the more quickly the sound echo attenuates. *SUI* pitch (frequency f_1) and *GU* pitch (frequency f_2) both have the same property. For a circular cylinder ($\psi = 90^\circ$), attenuation exponential β is the minimum. This is why sound echo of an ancient Chinese bell attenuates more quickly than a round bell, which is necessary to musical instruments, just because of its lens-shaped cross section as shown in Fig. 1.

In Table IV, we can see that ψ controls the short duration of sound. According to (47) and Table IV, the sound of the first tone decreases by 10 times (as $L/L_0 = 0.1$) after it spends approximately 4.9, 3.0, 1.8, and 1.2 s for $\psi = 85^\circ, 75^\circ, 65^\circ$, and 55° , respectively, and the sound of the second tone decreases by 10 times (as $L/L_0 = 0.1$) after it spends approximately 3.9, 1.9, 1.1, and 0.9 s for $\psi = 85^\circ, 75^\circ, 65^\circ$, and 55° , respectively.

V. CONCLUSIONS

Although the model proposed by the author is crude, we get some insight into the acoustics of an ancient Chinese

TABLE III. f_2/f_1 for different ψ of two kinds of double-circular arches No. 1 and No. 2. The more oblate a double-circular arch is, the larger the difference between the first and the second frequency.

ψ	90°	85°	80°	75°	70°	65°	60°	55°	50°
f_2/f_1	1.00	1.02	1.05	1.09	1.13	1.18	1.24	1.31	1.39

TABLE IV. Attenuation exponential β (1/s) for different ψ of the cylinder with the lens-shaped cross section like a double-circular arch No. 2 in Table II. The more oblate the cylinder is, the more quickly the sound echo attenuates.

ψ	90°	85°	80°	75°	70°	65°	60°	55°
β for f_1	0.42	0.47	0.60	0.77	0.99	1.27	1.60	1.90
β for f_2	...	0.59	0.85	1.20	1.65	2.15	2.57	2.60

bell. The lens-shaped cross section is an important factor for the two-tone bell. By a vibrating double-circular arch, we can explain theoretically two tones of an ancient Chinese bell. By acoustics of a vibrating cylinder with the lens-shaped cross section like a double-circular arch, we can explain the short duration of sound of an ancient Chinese bell. The numerical results of the model in this paper predict that the more oblate the lens-shaped cross section of Chinese bell is, the more quickly the sound attenuates.

However, it is difficult to model the acoustics of an ancient Chinese bell strictly because a three-dimensional problem on vibration and acoustics should be considered. To understand the ancient Chinese bell completely, we should appeal to numerical methods on the vibration and acoustics in three-dimensions.

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